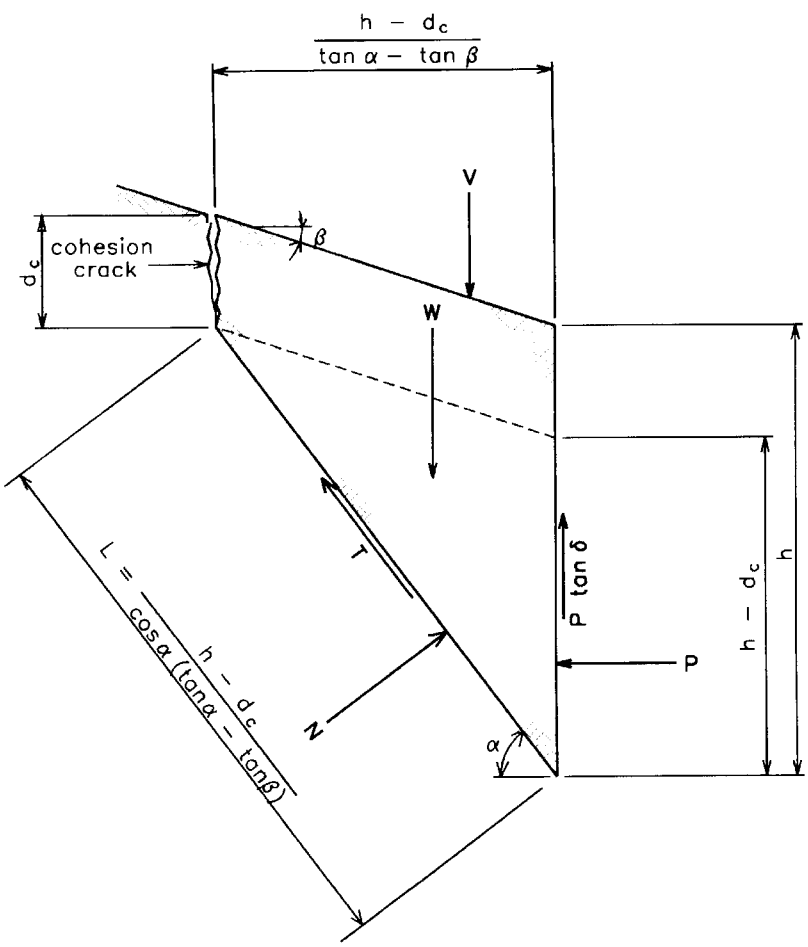
Appendix E
Derivations

Derivation of Equations to Determine Critical Slip Angle for Driving-Side Wedge with Wall Friction on the Vertical Face

The equations are derived for the limit equilibrium condition. The figure below shows the forces (including wall friction on the vertical face) acting on a driving-side wedge.



WEDGE FORCES

For equilibrium to exist:

$$N = (W + V)\cos\alpha + P\sin\alpha - P\tan\delta\cos\alpha$$
$$T = (W + V)\sin\alpha - P\cos\alpha - P\tan\delta\sin\alpha$$
$$T \text{ must also } = N\tan\phi + cL$$
$$T = N\tan\phi + cL = (W + V)\tan\phi\cos\alpha + P\tan\phi\sin\alpha - P\tan\delta\tan\phi\cos\alpha + cL$$

Equating the two expressions for T, dividing them by cos α, and solving for P, the following is obtained:

$$P = \frac{(W + V)(\tan\alpha - \tan\phi) - \dfrac{cL}{\cos\alpha}}{(1 - \tan\delta\tan\phi) + (\tan\delta + \tan\phi)\tan\alpha}$$

where

W = weight of soil in wedge

V = strip surcharge

$$W = \frac{\gamma\,(h^2 - d_c^2)}{2\,(\tan\alpha - \tan\beta)}, \quad \gamma = \text{unit weight of soil}$$

$$L = \frac{h - d_c}{\cos\alpha\,(\tan\alpha - \tan\beta)}$$

substituting the above values for W and L into the equation for P:

$$P = \frac{\left[\dfrac{\gamma\,(h^2 - d_c^2)}{2\,(\tan\alpha - \tan\beta)} + V\right](\tan\alpha - \tan\phi) - \dfrac{c\,(h - d_c)}{\cos^2\alpha\,(\tan\alpha - \tan\beta)}}{(1 - \tan\delta\tan\phi) + (\tan\delta + \tan\phi)\tan\alpha}$$

$$\textit{Note that}: \frac{1}{\cos^2\alpha} = \sec^2\alpha = 1 + \tan^2\alpha$$

divide both sides of this equation by:

$$\frac{\gamma\,(h^2 - d_c^2)}{2}, \quad \text{also substitute } 1 + \tan^2\alpha \text{ for } \frac{1}{\cos^2\alpha}$$

to obtain:

$$\frac{2P}{\gamma(h^2 - d_c^2)} = \frac{(\tan\alpha - \tan\phi) + \dfrac{2V}{\gamma(h^2 - d_c^2)}(\tan\alpha - \tan\beta)(\tan\alpha - \tan\phi) - \dfrac{2c(1 + \tan^2\alpha)}{\gamma(h + d_c)}}{(\tan\alpha - \tan\beta)[1 - \tan\delta\tan\phi + (\tan\delta + \tan\phi)\tan\alpha]} = \frac{m}{n}$$

combining terms the above equation becomes:

$$\frac{m}{n} = \frac{(\tan\alpha - \tan\phi) + \dfrac{2V}{\gamma(h^2 - d_c^2)}[\tan^2\alpha - (\tan\beta + \tan\phi)\tan\alpha + \tan\beta\tan\phi] - \dfrac{2c(1 + \tan^2\alpha)}{\gamma(h + d_c)}}{(\tan\delta + \tan\phi)\tan^2\alpha + [1 - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)]\tan\alpha - \tan\beta(1 - \tan\delta\tan\phi)}$$

The necessary condition for P to be either a maximum or a minimum is that the derivative of m/n, with respect to α, be equal to zero. The derivative of m/n is:

$$\frac{d\ (m/n)}{d\alpha} = \frac{n\left(\frac{dm}{d\alpha}\right) - m\left(\frac{dn}{d\alpha}\right)}{n^2} = 0$$

from this it can be seen that if both sides of the equation are multiplied by n^2 the maxima-minima condition becomes:

$$n\ \frac{dm}{d\alpha} - m\ \frac{dn}{d\alpha} = 0$$

After differentiating this equation, it will be found that all terms are multiplied by $\sec^2\ \alpha$ ($d\tan\alpha/d\alpha$). For simplification, both sides of the equation will be divided (at the start) by $\sec^2\alpha$ to eliminate it.

Since all of the terms in the numerator (m) are over a common denominator, differentiation will be done separately for each numerator term; they will be combined at the end to furnish a complete solution. The derivative dn/d'' is constant for all terms.

$$\frac{dn}{d\alpha} = 2\ (\tan\delta + \tan\phi)\ \tan\ \alpha + 1 - \tan\delta\ \tan\phi - \tan\beta\ (\tan\delta + \tan\phi)$$

<u>The "W" term (tan α - tan ϕ)</u>:

$$\frac{dm}{d\alpha} = 1$$

$$n\ \frac{dm}{d\alpha} = (\tan\delta + \tan\phi)\tan^2\alpha + (1 - \tan\delta\tan\phi)\tan\alpha - \tan\beta(\tan\delta + \tan\phi)\tan\alpha - \tan\beta(1 - \tan\delta\tan\phi)$$

$$m\ \frac{dn}{d\alpha} = 2(\tan\delta + \tan\phi)\ \tan^2\alpha + (1 - \tan\delta\tan\phi)\ \tan\alpha - \tan\beta(\tan\delta + \tan\phi)\ \tan\alpha$$
$$- 2\tan\phi\ (\tan\delta + \tan\phi)\tan\alpha - \tan\phi(1 - \tan\delta\ \tan\phi) + \tan\beta\tan\phi\ (\tan\delta + \tan\phi)$$

$$n\frac{dm}{d\alpha} - m\frac{dn}{d\alpha} = -(\tan\delta + \tan\phi)\tan^2\alpha + 2\tan\phi(\tan\delta + \tan\phi)\tan\alpha + \tan\phi - \tan\beta - (\tan\delta + \tan\beta)\tan^2\phi = 0$$

<u>The "V" term</u>:

$$\frac{dm}{d\alpha} = \left[\frac{4V}{\gamma\ (h^2 - d_c^2)}\right]\tan\alpha - \left[\frac{2V}{\gamma\ (h^2 - d_c^2)}\right](\tan\beta + \tan\phi)$$

$$n\frac{dm}{d\alpha} = \left[\frac{4V}{\gamma(h^2-d_c^2)}\right](\tan\delta+\tan\phi)\tan^3\alpha+\left[\frac{4V}{\gamma(h^2-d_c^2)}\right][1-\tan\delta\tan\phi-\tan\beta(\tan\delta+\tan\phi)]\tan^2\alpha$$
$$-\left[\frac{4V}{\gamma(h^2-d_c^2)}\right]\tan\beta(1-\tan\delta\tan\phi)\tan\alpha - \left[\frac{2V}{\gamma(h^2-d_c^2)}\right](\tan\beta+\tan\phi)(\tan\delta+\tan\phi)\tan^2\alpha$$
$$-\left[\frac{2V}{\gamma\ (h^2-d_c^2)}\right](\tan\beta+\tan\phi)[1-\tan\delta\ \tan\phi-\tan\beta\ (\tan\delta+\tan\phi)]\ \tan\alpha$$
$$+\left[\frac{2V}{\gamma(h^2-d_c^2}\right]\tan\beta\ (\tan\beta+\tan\phi)(1-\tan\delta\ \tan\phi)$$

$$m\frac{dn}{d\alpha} = \left[\frac{4V}{\gamma(h^2-d_c^2)}\right][(\tan\delta+\tan\phi)\tan^3\alpha-(\tan\delta+\tan\phi)(\tan\beta+\tan\phi)\tan^2\alpha+\tan\beta\tan\phi(\tan\delta+\tan\phi)\tan\alpha]$$
$$+\left[\frac{2V}{\gamma(h^2-d_c^2)}\right][1-\tan\delta\ \tan\phi-\tan\beta\ (\tan\delta+\tan\phi)]\ [\tan^2\alpha-(\tan\beta+\tan\phi)\ \tan\alpha]$$
$$+\left[\frac{2V}{\gamma(h^2-d_c^2)}\right][1-\tan\delta\ \tan\phi-\tan\beta\ (\tan\delta+\tan\phi)]\ \tan\beta\ \tan\phi = 0$$

$$n\frac{dm}{d\alpha} - m\frac{dn}{d\alpha} = \left[\frac{2V}{\gamma\ (h^2-d_c^2)}\right](1+\tan^2\phi)\ \tan^2\alpha - \left[\frac{4V}{\gamma\ (h^2-d_c^2)}\right]\tan\beta(1+\tan^2\phi)\ \tan\alpha$$
$$+\left[\frac{2V}{\gamma\ (h^2-d_c^2)}\right]\tan^2\beta\ (1+\tan^2\phi) = 0$$

<u>The “c” term:</u>

$$m = -\frac{2c}{\gamma\ (h+d_c)} - \frac{2c\ \tan^2\alpha}{\gamma\ (h+d_c)}, \qquad \frac{dm}{d\alpha} = -\frac{4c\ \tan\alpha}{\gamma\ (h+d_c)}$$

$$\frac{dn}{d\alpha} = 2(\tan\delta+\tan\phi)\ \tan\alpha + [1-\tan\delta\ \tan\phi-\tan\beta\ (\tan\delta+\tan\phi)]$$

$$n\frac{dm}{d\alpha} = -\left[\frac{4c}{\gamma\ (h+d_c)}\right](\tan\delta+\tan\phi)\ \tan^3\alpha$$
$$-\left[\frac{4c}{\gamma\ (h+d_c)}\right][1-\tan\delta\tan\phi-\tan\beta(\tan\delta+\tan\phi)]\ \tan^2\alpha$$
$$+\left[\frac{4c}{\gamma\ (h+d_c)}\right][\tan\beta\ (1-\tan\delta\ \tan\phi)]\ \tan\alpha$$

$$m\,\frac{dn}{d\alpha} = -\left[\frac{4c}{\gamma\,(h + d_c)}\right](\tan\delta + \tan\phi)\,\tan^3\alpha$$
$$-\left[\frac{2c}{\gamma\,(h + d_c)}\right][1 - \tan\delta\tan\phi - \tan\beta\,(\tan\delta + \tan\phi)]\,\tan^2\alpha$$
$$-\left[\frac{4c}{\gamma\,(h + d_c)}\right](\tan\delta + \tan\phi)\,\tan\alpha$$
$$-\left[\frac{2c}{\gamma\,(h + d_c)}\right][1 - \tan\delta\tan\phi - \tan\beta\,(\tan\delta + \tan\phi)\,]$$

$$n\,\frac{dm}{d\alpha} - m\,\frac{dn}{d\alpha} = -\left[\frac{2c}{(h + d_c)}\right][\,1 - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)]\,\tan^2\alpha$$
$$+\left[\frac{4c}{\gamma\,(h + d_c)}\right][\tan\beta\,(1 - \tan\delta\tan\phi) + \tan\delta + \tan\phi\,]\,\tan\alpha$$
$$+\left[\frac{2c}{\gamma\,(h + d_c)}\right][1 - \tan\delta\tan\phi - \tan\delta\tan\beta - \tan\beta\tan\phi\,] = 0$$

<u>Combining "*W*", "*V*", and "*c*" terms:</u>

$$-\,[\tan\phi + \tan\delta - \left[\frac{2V}{\gamma(h^2 - d_c^2)}\right](1 + \tan^2\phi) + \left[\frac{2c}{\gamma(h + d_c)}\right](1 - \tan\delta\tan\phi - \tan\beta[\tan\delta + \tan\phi])]\,\tan^2\alpha$$
$$+\,[\,2\tan\phi\,(\tan\delta + \tan\phi) - \left[\frac{4V}{\gamma\,(\,h^2 - d_c^2\,)}\right]\tan\beta(\,1 + \tan^2\phi\,)$$
$$+\left[\frac{4c}{\gamma\,(\,h + d_c\,)}\right](\,\tan\beta + \tan\phi + \tan\delta\,[\,1 - \tan\beta\tan\phi\,]\,)\,]\,\tan\alpha$$
$$+\,[\,\tan\phi - \tan\beta - (\,\tan\delta + \tan\beta\,)\tan^2\phi + \left[\frac{2V}{\gamma(\,h^2 - d_c^2\,)}\right]\tan^2\beta\,(\,1 + \tan^2\phi)$$
$$+\left[\frac{2c}{\gamma\,(\,h + d_c\,)}\right][\,1 - \tan\delta\tan\phi - \tan\beta\,(\,\tan\delta + \tan\phi\,)\,] = 0$$

In order to make the above equation less cumbersome, let:

$$1 - \tan\delta\tan\phi - \tan\beta\,(\,\tan\delta + \tan\phi\,) = r$$
$$\tan\beta + \tan\phi + \tan\delta\,(\,1 - \tan\beta\tan\phi\,) = s$$
$$\tan\phi - \tan\beta - (\,\tan\delta + \tan\beta\,)\tan^2\phi = t$$

Then the equation becomes:

$$
\begin{aligned}
&-\left[\tan\phi + \tan\delta - \left(\frac{2V}{\gamma\,(h^2 - d_c^2)}\right)(1 + \tan^2\phi) + \left(\frac{2c}{\gamma\,(h + d_c)}\right) r\right]\tan^2\alpha \\
&+\left[2\tan\phi\,(\tan\delta + \tan\phi) - \left(\frac{4V}{\gamma\,(h^2 - d_c^2)}\right)\tan\beta\,(1 + \tan^2\phi) + \left(\frac{4c}{\gamma\,(h + d_c)}\right) s\right]\tan\alpha \\
&+\left[t + \left(\frac{2V}{\gamma\,(h^2 - d_c^2)}\right)\tan^2\beta\,(1 + \tan^2\phi) + \left(\frac{2c}{\gamma\,(h + d_c)}\right) r\right] = 0
\end{aligned}
$$

Denoting the coefficient of $\tan^2\alpha$ as $-A$, the coefficient of $\tan\alpha$ as AC_1, and the constant as AC_2 we have the following quadratic equation:

$$-A\tan^2\alpha + AC_1\tan\alpha + AC_2 = 0$$

Dividing this by $-A$ we obtain:

$$\tan^2\alpha - C_1\tan\alpha - C_2 = 0$$

The solution for $\tan\alpha$ then is:

$$\tan\alpha = \frac{C_1 + \sqrt{C_1^2 + 4C_2}}{2}$$

alternately

$$\alpha = \tan^{-1}\left(\frac{C_1 + \sqrt{C_1^2 + 4C_2}}{2}\right)$$

where

$$A = \tan\phi + \tan\delta - \left(\frac{2V}{\gamma\,(h^2 - d_c^2)}\right)(1 + \tan^2\phi) + \left(\frac{2c}{\gamma\,(h + d_c)}\right) r$$

$$C_1 = \frac{2\tan\phi\,(\tan\delta + \tan\phi) - \left(\frac{4V}{\gamma\,(h^2 - d_c^2)}\right)\tan\beta\,(1 + \tan^2\phi) + \left(\frac{4c}{\gamma\,(h + d_c)}\right) s}{A}$$

$$C_2 = \frac{t + \left(\frac{2V}{\gamma\,(h^2 - d_c^2)}\right)\tan^2\beta\,(1 + \tan^2\phi) + \left(\frac{2c}{\gamma\,(h + d_c)}\right) r}{A}$$