APPENDIX B

THERMAL MODELS FOR COMPUTING FREEZE AND THAW DEPTHS[1]

B-1. Analytical Solutions.

a. The Oxford University Press publication, *Conduction of Heat in Solids*,[2] presents solutions to many one-, two- and three-dimensional heat flow problems. Homogeneous isotropic materials are used in most solutions, but some solutions are presented for layered systems. U.S.G.S. Bulletin 1083-A[3] developed a one-dimensional technique for predicting the damping of a periodic surface temperature at different depths in a two- and three-layered soil system. Unidirectional heat flux was considered. U.S.G.S. Bulletin 1052-B[4] developed a method for estimating the three-dimensional thermal regime in a homogeneous isotropic soil beneath a heated structure. None of these techniques considers phase change of the soil moisture. Neglect of the effects of latent heat of fusion of the soil moisture normally does not cause substantial error in prediction of frost depths in soil of low water content. Differences between actual and computed thaw depths increase rapidly with increasing water content because of the increased volumetric heat capacity and greater latent heat of the wetter soil.

b. Several emprical and semi-empirical equations have been developed that consider latent heat of fusion of the soil. the Stefan equation was originally developed to calculate the thickness of ice on a calm body of water (isothermal at the freezing temperature, 32°F) expressed as:

$$X_i = \sqrt{48K_iF/L_i} \quad \text{(eq B-1)}$$

where

X_i = ice thickness (ft)

K_i = thermal conductivity of ice (Btu/ft hr °F)

F = freezing index (degree-days)

L_i = volumetric latent heat of fusion of ice (Btu/ft³).

The Stefan equation has been modified by many individuals and agencies, and many similar equations have been developed. Some of the equations use functions or initial conditions slightly different from those used in the original Stefan model. The most widely used equation to estimate seasonal freeze and thaw depths is the modified Berggren equation. Application of this equation, given below, has been very widespread in North America. USACRREL Special Report 122[5] developed a computer program for calculating freeze and thaw depths in layered systems using this equation:

$$X = \sqrt{48KnF/L} \text{ or } X = \lambda\sqrt{48KnI/L} \quad \text{(eq B-2)}$$

where

X = depth of freeze or thaw (ft)

K = thermal conductivity of soil (Btu/ft hr °F)

L = volumetric latent heat of fusion (Btu/ft³)

n = conversion factor from air index to surface index (dimensionless)

F = air freezing index (degree-days)

I = air thawing index (degree-days)

λ = coefficient that considers the effect of temperature changes within the soil mass. It is a function of the freezing (or thawing) index, the mean annual temperature and the thermal properties of the soils.

[1] The documents mentioned in this appendix are sources for additional information and are found in the bibliography.

[2] Carslaw and Jaeger, 1959.

[3] Lachenbruch, 1959.

[4] Lachenbruch, 1957.

[5] Aitken and Berg, 1968.

c. An equation very similar to the Stefan equation is used in the USSR to calculate the "standard" depth of freezing for foundation designs. Many other closed-form analytical techniques are also used in the USSR.

B-2. Graphical and analog methods.

a. Graphical methods have been used to estimate depths of freeze and thaw. The flow net technique can be used to estimate steady-state temperature conditions. National Research Council of Canada, Technical Paper No. 163,[6] presents a graphical means to determine temperature in the ground under and around natural and engineering structures lying directly on the ground surface.

b. Analog techniques are also used to estimate freeze and thaw depths. Table B-1 shows thermal, fluid and electrical analogies. Electric analog computers are available, cost relatively little and are reasonably simple to use. The primary disadvantage of these machines are that reprogramming is normally necessary for each problem and complex geometries are difficult to simulate adequately. Hydraulic analogs are also available. The primary disadvantages of these computers are their complex tubing systems, space requirements, and the necessity to thoroughly clean and reconstruct them for each problem. At any instant of time, however, hydraulic analogs graphically show the temperature distribution.

[6] Brown, 1963.

B-3. Numerical techniques.

a. Because of the widespread availability of electronic digital computers, their application to numerical solutions of the continuity equation is commonplace. Numerical procedures are approximations to the partial differential equation; however, they are much more accurate and versatile in solving complex transient heat flow problems than are the analytical techniques. Many computer programs allow flexible definition of boundary and initial conditions for both one- and two-dimensional problems. General background information on the *finite difference methods* available to solve heat flow problems are discussed in International Textbook Company's *Heat Transfer Calculations by Finite Differences.*[7] Since rectangular elements are normally used, complex geometries are difficult to simulate accurately unless small element sizes are used.

b. Use of the *finite element technique* is also widespread (see *The Finite Element Method in Engineering Science*[8] or *The Finite Element Method in Structural and Continuum Mechanics,*[9] both from McGraw-Hill). Elements of various shapes can be used with this technique; but the triangular shape is commonly used for two-dimensional problems. Complex boundary geometries can be more closely

[7] Dusinberre, 1961.
[8] Zienkiewicz, 1967.
[9] Zienkiewicz, 1971.

Table B-1. Thermal, fluid and electric analogs (U.S. Army Corps of Engineers).

		Medium					
Item		*Thermal*		*Fluid*		*Electric*	
A - Variables	(1)	Heat	μ	Volume	S	Charge density	ρ
	(2)	Heat flux	$\vec{q}$	Flow	$\vec{Q}$	Current density	$\vec{j}$
	(3)	Temperature	T	Head	$\overline{H}$	Voltage	e
B - Principles:							
Continuity	(1)	$\frac{\partial \mu}{\partial t} + \vec{\nabla} \cdot \vec{q} = 0$		$\frac{\partial S}{\partial t} + \vec{\nabla} \cdot \vec{Q} = 0$		$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$	
Conductivity	(2)	$\vec{q} = -k\vec{\nabla}T$		$\vec{Q} = -k\vec{\nabla}H$		$\vec{j} = -\sigma\vec{\nabla}e$	
Capacitance	(3)	$d\mu = CdT$		$dS = AdH$		$\rho dV = Cde$	

simulated using finite element procedures. For multidimensional heat flow problems, the finite element procedure is frequently more efficient, i.e., it requires less computer time than the finite difference technique.

c. Many flexible computer programs exist that simulate heat conduction and phase change in soils. Each has its own particular data requirements, computational capabilities, and acquisition costs and restraints. USACRREL has completed and documented a model that may be useful in solving many analytical problems related to construction in the Arctic and Subarctic; other models are under development. Contact HQ (DAEN-ECE-G) or HQ AFESC for assistance in selecting an appropriate model.