Chapter 3
Time History Numerical Solution Techniques

3-1. Introduction

a. Numerical solutions of equations of motion for structures are divided into two methods: direct integration and mode superposition. In direct integration the equations of motion are integrated directly using a numerical step-by-step procedure, without transforming the equations into a different form. In mode superposition method the equations of motion are first transformed into a more effective form (modal forms) before they are solved using either a step-by-step integration in the time domain or by application of the frequency-domain procedures. Only a brief description of these procedures as applied to the linear time-history analysis are presented in this chapter. A more detailed description of the methods can be found in Ebeling, Green, and French (1997) and other cited references. The chapter is divided into two sections: Section I and Section II. Section I begins with a general description of the equations of motion followed by presentation of several time-domain solution procedures commonly used in the earthquake engineering. Section II is devoted to solution of the equations of motion in the frequency domain. It begins with a preview and conversion of the seismic input into the frequency domain, followed by introduction of the frequency response functions, and finally computation of the structural response first in the frequency domain and then converting back to the time domain.

Section I
Time Domain Solutions

3-2. Equations of Motion

a. The equations of motion for a hydraulic structure are formulated from the equilibrium of the effective forces associated with each of its degrees of freedom (Clough and Penzien, 1993). In a matrix form these equations may be expressed by

$$\mathbf{m}\,\ddot{\mathbf{u}}(t)+\mathbf{c}\,\dot{\mathbf{u}}(t)+\mathbf{k}\,\mathbf{u}(t)=\mathbf{p}(t) \qquad (3\text{-}1)$$

where **m**, **c**, and **k** are the mass, damping, and stiffness matrices, respectively; $\mathbf{u}(t)$, $\dot{\mathbf{u}}(t)$, and $\ddot{\mathbf{u}}(t)$ are respectively nodal displacement, velocity, and acceleration vectors; and **p**(t) is the effective load vector. For hydraulic structures the coefficient matrices and the effective load vector also include contributions from interaction with the water and the foundation rock. However, depending on the type of finite-element formulation used, the interaction effects are treated as described below.

b. In standard finite-element formulation (2-11) with massless foundation and incompressible water the mass matrix **m** contains both the structure mass and the added mass of water. The matrix **k** is the combined stiffness for the structure and the foundation. The effective load vector **p**(t) resulting from the earthquake ground shaking, includes inertia loads due to the mass of the structure and of the water.

c. In substructuring finite-element formulation (2-10), the interaction with water is generally represented by hydrodynamic forces that are included as part of the effective load vector. The damping and stiffness are represented in the form of dynamic stiffness or impedance matrix that also includes interacting forces with the foundation. Since foundation impedance and hydrodynamic forces are frequency dependent, the substructuring formulation is usually developed in the frequency domain and solved as described in Section II of this chapter.

3-3. Direct Integration

a. General. In direct integration the equations of motion in (3-1) are directly integrated using a step-by-step numerical procedure without prior transformation of the equations to a different form. The step-by-step integration procedures provide an approximate solution at *n* discrete time intervals 0, Δt, $2\Delta t$, $3\Delta t$, …, t, $t+\Delta t$, …T, where T is duration of the input motion or loading and $\Delta t = T/n$. Many numerical integration procedures have been developed. However, only a very brief summary of the most common methods are presented to show how they are applied in seismic analysis of hydraulic structures. Direct integration methods are generally classified as either *explicit* or *implicit*. In general, linear time-history analyses of hydraulic structures described in Chapter 2 are conducted using implicit methods. However, explicit methods are briefly discussed here because they are included in some commercial computer programs and the reader may use them in solution of certain problems.

b. Explicit Methods. The basic concept common to most explicit methods is to write the equations of motion for the beginning of the time step, approximate the initial velocity and acceleration terms by finite-difference expressions, and then solve for response at the end of time step. This way the response values calculated in each step depend only on quantities obtained in the preceding step. Therefore, the numerical process proceeds directly from one step to next. Explicit methods are very convenient, but they are only conditionally stable and will "blow up" if time step is not sufficiently small.

(1) The central Difference Method. The central difference method is a very simple explicit method that uses the following finite-difference expressions for approximation of the initial velocity and acceleration terms (Clough and Penzien 1993, Bathe and Wilson 1976).

$$\ddot{\mathbf{u}}_t = \frac{1}{\Delta t^2}\left[\mathbf{u}_{t-\Delta t} - 2\mathbf{u}_t + \mathbf{u}_{t+\Delta t}\right] \tag{3-2}$$

$$\dot{\mathbf{u}}_t = \frac{1}{2\Delta t}\left(-\mathbf{u}_{t-\Delta t} + \mathbf{u}_{t+\Delta t}\right) \tag{3-3}$$

The displacement solution for time step t+Δt (i.e. $u_{t+\Delta t}$) is obtained by considering the equations of motion (i.e. (3-1)) at time step t

$$\mathbf{m}\ddot{\mathbf{u}}_t + \mathbf{c}\dot{\mathbf{u}}_t + \mathbf{k}\mathbf{u}_t = \mathbf{p}_t \tag{3-4}$$

Substituting (3-2) and (3-3) into (3-4), leads to

$$\left(\frac{1}{\Delta t^2}\mathbf{m} + \frac{1}{2\Delta t}\mathbf{c}\right)\mathbf{u}_{t+\Delta t} = \mathbf{p}_t - \left(\mathbf{k} - \frac{2}{\Delta t^2}\mathbf{m}\right)\mathbf{u}_t - \left(\frac{1}{\Delta t^2}\mathbf{m} - \frac{1}{2\Delta t}\mathbf{c}\right)\mathbf{u}_{t-\Delta t} \tag{3-5}$$

This equation shows that the stiffness matrix does not appear in the coefficient of $\mathbf{u}_{t+\Delta t}$, and thus factorization of the (effective) stiffness matrix will not be required. If mass and damping matrices are diagonal, even the matrices need not be assembled. In that case the solution is obtained at the element level which allows solving very large structural systems without substantial computer resources. These advantages make the central difference method convenient and efficient. However, the method is only conditionally stable and its stability depends on the choice of time step Δt. To obtain a stable solution, Δt must be $< T_n/\pi$, where T_n is the shortest natural period of the structural system. If the central difference method is used with a $\Delta t > T_n/\pi$ the solution will increase exponentially. This may not be a limiting factor for a SDOF system, because more than π or 3 steps are needed to adequately define one vibration cycle. However, the size of time step would be an issue for the MDOF systems where very short-period modes of vibration (compared with time step of input loading) are essential to the dynamic response of the system. In these situations extremely small time step would make the explicit methods undesirable.

c. Implicit Methods. In an implicit method the expressions for new values at $t+\Delta t$ use equilibrium equations at $t+\Delta t$, and thus include one or more values pertaining to that same step. Examples of implicit integration methods are presented below.

(1) Newmark-β Method. The Newmark-β method is a general step-by-step procedure with the following integration equations for the displacement and velocity at time step $t+\Delta t$ (Clough and Penzien 1993, Bathe and Wilson 1976).

$$\mathbf{u}_{t+\Delta t} = \mathbf{u}_t + \Delta t\,\dot{\mathbf{u}}_t + \Delta t^2\left[\left(\frac{1}{2}-\beta\right)\ddot{\mathbf{u}}_t + \beta\,\ddot{\mathbf{u}}_{t+\Delta t}\right] \tag{3-6}$$

$$\dot{\mathbf{u}}_{t+\Delta t} = \dot{\mathbf{u}}_t + \Delta t\left[(1-\gamma)\ddot{\mathbf{u}}_t + \gamma\,\ddot{\mathbf{u}}_{t+\Delta t}\right] \tag{3-7}$$

where β and γ are weighting factors and can be chosen to obtain optimum stability and accuracy.

(a) If $\beta = 1/4$ and $\gamma = 1/2$ the Newmark-β method is unconditionally stable (i.e. regardless of the size of the time step). In this case the acceleration within the time step Δt is constant and is usually referred to as the Newmark's constant-average acceleration scheme.

(b) If $\beta = 1/6$ and $\gamma = 1/2$ the Newmark-β method is identical to the linear acceleration method, in which the acceleration varies linearly within each time step. The linear acceleration method, however, is only conditionally stable. The linear acceleration method will be unstable unless $\Delta t/T_n \leq \sqrt{3}/\pi = 0.55$. This restriction has no effect in the analysis of SDOF systems because a shorter time step than $\Delta t/T_n = 0.55$ is needed for satisfactory representation of the dynamic response and input, but may require extremely short time step for analysis of MDOF systems having short periods of vibration.

(c) In general assuming a linear acceleration within each time step gives a better approximation of the dynamic response and provides more accurate results than the constant acceleration method.

(2) Wilson θ Method. For a general type of structure the period of the highest mode is related to the properties of the individual elements. An element with relatively small mass results in a very short period of vibration, with the effect of requiring an extremely short time step of integration, as discussed above. Although the unconditionally-stable constant acceleration method can be used in this situation, for accuracy reasons an unconditionally-stable linear acceleration method such as the Wilson θ-method (Clough and Penzien 1993, Bathe and Wilson 1976) is more desirable. The Wilson θ-method is based on the assumption that the acceleration varies linearly over an extended computational interval $\tau = \theta\Delta t$, *where* $\theta \geq 1$. For $\theta = 1$, the method reverts to the standard linear acceleration method, but for $\theta > 1.37$ it becomes unconditionally stable. However, the Wilson θ–method tends to damp out the higher modes and could produce large errors when contributions of higher modes are significant. Therefore, the use of this method is no longer recommended. Instead the Hilbert, Hughes and Taylor α method, described next, is now being implemented in many computer programs in recent years.

(3) The Hilbert, Hughes and Taylor α Method. The HHT-α method (Hughes 1997) is a generalization of the Newmark-β method and reduces to the Newmark-β method for $\alpha = 0$. The finite-difference equations for the HHT-α method are identical to those of the Newmark-β method (i.e. (3-6) and (3-7)). The equations of motion are modified, however, using a parameter α.

$$\mathbf{m}\ddot{\mathbf{u}}_{t+\Delta t} + (1+\alpha)\mathbf{c}\dot{\mathbf{u}}_{t+\Delta t} - \alpha\,\mathbf{c}\dot{\mathbf{u}}_t + (1+\alpha)\mathbf{k}\mathbf{u}_{t+\Delta t} - \alpha\,\mathbf{k}\mathbf{u}_t = (1+\alpha)\mathbf{f}_{t+\Delta t} - \alpha\,\mathbf{f}_t \qquad (3\text{-}8)$$

With $\alpha = 0$ the HHT-α method reduces to the constant acceleration method. If $-1/3 \le \alpha \le 0$, $\beta = (1-\alpha^2)/4$, and $\gamma = 1/2-\alpha$, the HHT-α method is second-order accurate and unconditionally stable. The HHT-α method is useful in structural dynamics simulations incorporating many degrees of freedom, and in which it is desirable to numerically attenuate (or dampen-out) the response at high frequencies. Decreasing α (below zero) decreases the response at frequencies above $1/(2\Delta t)$, provided that β and γ are defined as above.

3-4. Mode Superposition

a. The number of operations in the direct integration method is proportional to the number of time steps used. In general the use of direct integration may be considered effective when the response is required only for a relatively short duration. However, if the integration must be carried out for many time steps, it may be more effective to transform the equations of motion into a form for which the step-by-step integration is less costly. For this purpose the equations of motion for linear analysis are usually transformed into the eigenvectors or normal-coordinate system. Applying the normal-coordinate transformation in accordance with Clough and Penzien (1993) to Equation (3-1) leads to the following decoupled equation of motion for individual modes

$$M_n\ddot{Y}_n(t) + C_n\dot{Y}(t) + K_nY_n(t) = P_n(t) \qquad (3\text{-}9)$$

where the modal coordinate mass, damping, stiffness, and load are defined as follows:

$$M_n = \phi_n^T\mathbf{m}\phi_n \qquad (3\text{-}10a)$$

$$C_n = \phi_n^T\mathbf{c}\phi_n \qquad (3\text{-}10b)$$

$$K_n = \phi_n^T\mathbf{k}\phi \qquad (3\text{-}10c)$$

$$P_n = \phi_n^T\mathbf{p}(t) \qquad (3\text{-}10d)$$

The modal Equation (3-9) may also be expressed in the following form

$$\ddot{Y}_n(t) + 2\xi_n\omega_n\dot{Y}_n(t) + \omega_n^2Y_n(t) = \frac{P_n(t)}{M_n} \qquad (3\text{-}11)$$

where ξ_n is the modal damping ratio, and ω_n is the undamped natural frequency. Now the time integration can be carried out individually for each decoupled modal equation (3-11). This can be accomplished using any of the above integration schemes or by numerical evaluation of the Duhamel integral (Clough and Penzien, 1993)

$$Y_n(t) = \frac{1}{M_n\omega_n}\int_0^t P_n(\tau)\exp[-\xi_n\omega_n(t-\tau)]\sin\omega_{Dn}(t-\tau)\,d\tau \qquad (3\text{-}12)$$

where $\omega_{Dn} = \omega_n \sqrt{1-\xi_n^2}$ is the damped natural frequency. Having obtained the response of each mode Y_n (t) from Equation (3-12), the total displacement of the structure in the geometric coordinates can be computed using

$$\mathbf{u}(t) = \phi_1 Y_1(t) + \phi_2 Y_2(t) + \ldots + \phi_N Y_N(t) \tag{3-13}$$

b. In summary, the response analysis by mode superposition requires: 1) the solution of the eigenvalues and eigenvectors for transformation of the problem to the modal coordinates, 2) solution of the decoupled modal equilibrium equations (3-11) by the Duhamel integral or other integration schemes, and 3) superposition of the modal responses as expressed in (3-13) to obtain the total response of the structure.

c. In the linear time-history analysis the choice between the direct method and mode superposition is decided by effectiveness of the methods and whether a few modes of vibration can provide accurate results or not. The solutions obtained using either method are identical with respect to errors inherit in the time integration schemes and round-off errors associated with computer analysis.

3.5 Stability and Accuracy Considerations

a. The aim of numerical integration of the equations of motion is to obtain stable and accurate approximation of the dynamic response with minimal computational effort. An integration method is unconditionally stable if solution for any initial conditions will not grow without bound for any time step Δt. The method is said to be only conditionally stable if the aforementioned stability condition holds when $\Delta t/T$ is smaller than a critical value. On this basis, it is clear that if a conditionally stable method such as the Newmark's linear acceleration method is employed, the time step Δt should be less than the value specified in 3-3*c*(1)(b). While an unconditionally stable method such as the Newmark's constant acceleration has no restriction on the size of the time step for stability consideration, the time step should still be small enough so that the method yields an accurate and effective solution. In general stability and accuracy of any method can be improved by reducing the size of the time step.

b. The main factor in selecting a step-by-step method is efficiency, which concerns with the computational effort needed to achieve desired level of accuracy. Accuracy alone is not a good criterion for method selection. This is because any desired level of accuracy can be achieved by any method if the time step is made adequately short. In any case the time step should be short enough to provide adequate definition of the loading and the response history. A high frequency load or response cannot be described by long time steps.

c. The integration time step should be selected based on the frequency content of the applied load and the highest significant frequency (shortest significant period) of vibration of the structure. If the load history is relatively simple, the choice of the time step will depend essentially on the shortest significant period of vibration. In general, using a time step $\Delta t \le T_p/10$ will give reliable results. Here T_p is the lowest period of vibration that will contribute significantly to the dynamic response. Considering that an earthquake tends to excite mainly a few lower modes of vibration, T_p need not be very short. For example a $\Delta t = 0.01$ can adequately define an earthquake acceleration digitized at intervals of 0.01 sec and provide accurate results for periods of vibration up to 0.1 sec. Note that even though such an earthquake acceleration contains frequencies up to Nyquist frequency $f = 1/2\Delta t = 50$ Hz, the significant input energy is usually confined to frequencies less than about 10 Hz (0.1 sec). Thus selecting a short time step than 0.01 sec to account for higher modes may not change the response significantly, because the earthquake input energy to excite such higher modes is expected to be negligible. If there is any doubt about the accuracy of the results, a second analysis can be made with a time step reduced by one-half to ensure that the errors produced by numerical integration are acceptable.

Section II
Frequency Domain Solutions

3-6. Preview

An alternative approach to solving the modal equations of motion (3-9), instead of step-by-step integration in the time domain, is to make use of the frequency domain analysis procedures. The frequency domain analysis is especially convenient when the equation of motion contains frequency dependent stiffness and damping parameters, as in the case of structure-foundation and fluid-structure interaction problems (Clough and Penzien 1993). Only an outline of the frequency domain analysis is presented here. A frequency domain solution consists of three phases as described below and illustrated in Figure 3-1.

a. The first phase involves conversion of the applied loading from the time domain to the frequency domain by means of the Fourier transform procedure. This process replaces the loading amplitude values expressed at a sequence of time steps by complex values that express the harmonic load amplitudes at a specified sequence of frequencies. These complex values may be interpreted as the frequency-domain expression of the loading.

b. In the second phase the structure responses for a specified sequence of given frequencies are characterized by the complex frequency response functions, which express the harmonic response amplitudes of the structure due to unit harmonic loading. In other words, the frequency response function of a system is the ratio of response amplitude to input amplitude when the input is a complex exponential or sinusoid. When the complex frequency response functions are multiplied by the harmonic input amplitudes obtained in Step 1, the results give the frequency-domain expression of the structure response for the specified loading.

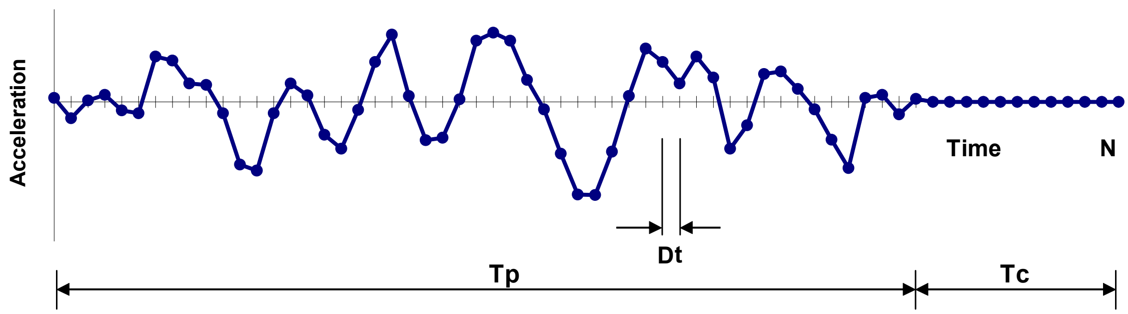
c. In the final phase of the analysis, the frequency domain response obtained in Step 2 is converted back to the time domain by means of the inverse Fourier transform procedure. Once the time-domain structure response of interest, usually displacements, have been determined in this manner, other response quantities such as stresses and section forces and moments are evaluated using the standard relationships between displacements and these quantities.

3-7. Transfer of Acceleration Time-History to the Frequency Domain

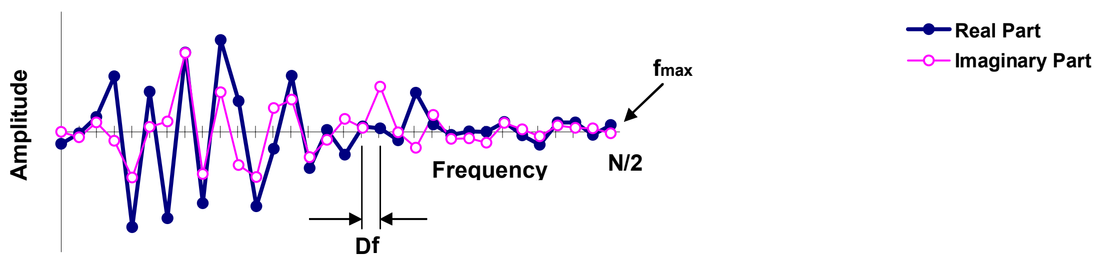
a. The first step in the frequency-domain response analysis is an understanding that an acceleration time-history $\ddot{x}_{ground}(t)$ can be represented by Fourier series using the discrete Fourier transform. Consider for example an $\ddot{x}_{ground}(t)$ digitized at a time-step Δt = 0.01 sec with a total duration t_p = 40.96 sec. The total number of points N, equally spaced in time, is 4096 ($N = t_p/\Delta t$). The trigonometric form of the Fourier series representation of the acceleration time-history is given by:

$$\ddot{x}_{ground}(t) = a_o + \sum_{n=1}^{infinity} a_n \cos(\omega_n t) + b_n \sin(\omega_n t) \tag{3-14}$$

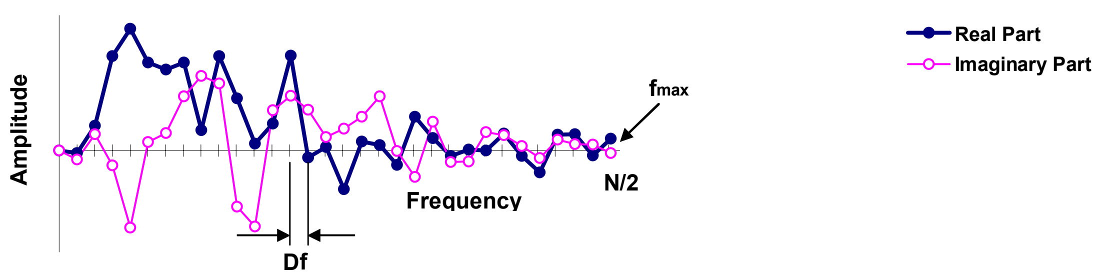
where a_0, a_n, and b_n are *Fourier coefficients*. All Fourier coefficients a_0, a_n, and b_n are constants for a given time-history. Note that the Fourier series is simply a summation of simple harmonic sine and cosine functions. The general trigonometric form of the Fourier series assumes a periodic function of period T_p, equal to duration t_p, and $\omega_n = 2\pi n/T_p$. These circular frequencies ω_n are not arbitrary but equally spaced at a circular frequency increment $\Delta\omega = 2\pi/T_p$ $(=2\pi/t_p)$.



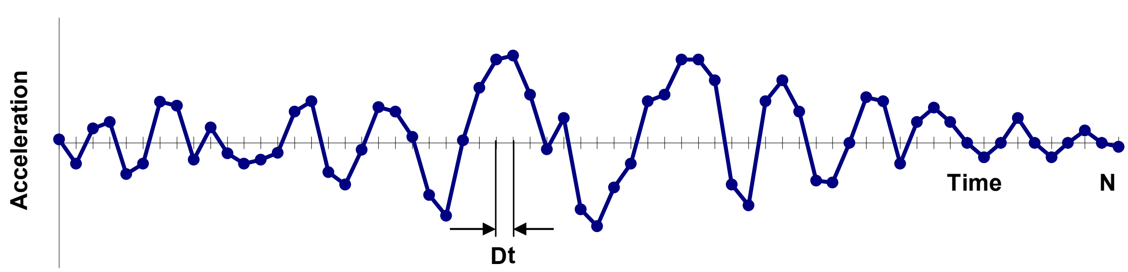
a) Earthquake acceleration



b) Complex-valued Fourier transform of earthquake acceleration



c) Complex-valued frequency response function of the structure computed for unit harmonic input



d) Time history of response computed by inverse Fourier transform of the product of b and c above

Figure 3-1. Phases of dynamic response computation in frequency domain and relationships between response parameters *b.* In practice the Fourier series of $\ddot{x}_{ground}(t)$ is frequently expressed in complex exponential form and computed using the computerized Fast Fourier Transform (FFT):

$$\ddot{x}_{ground}(t) = Re \sum_{s=0}^{N/2} \ddot{X}_s \exp\left(i\, \omega_s\, t \right) \qquad (3\text{-}15)$$

where *Re* designates the real component of the complex series and

$\ddot{X}_s = (N/2+1)$ complex Fourier amplitudes

ω_s = the circular frequency of each of (N/2 + 1) harmonic

and

$$i = \sqrt{-1} \qquad (3\text{-}16)$$

The circular frequency of each harmonic ω_s is given by:

$$\omega_s = \frac{2\,\pi\, s}{T_P} = \frac{2\,\pi\, s}{t_p} \quad for\ s = 0, 1, 2, \ldots, N/2 \qquad (3\text{-}17)$$

Note that for a given time-history each complex Fourier coefficient $\ddot{X}_s$ is a constant value corresponding to each circular frequency ω_s and comprises of real and imaginary parts. Use of the FFT is based on letting $N = 2^j$ (e.g., N = …, 512, 1024, …, 4096, etc.), following the Cooley and Tukey (1965) algorithm for computers.

3-8. Frequency Response Functions

a. *Definition*. The frequency response function of a system is the ratio of response (output) amplitude to input amplitude when the input is a complex exponential or sinusoid. For the input force set to

$$p(t) = Pe^{i\omega t} \qquad (3\text{-}18)$$

where F is the input amplitude, the output displacement x will be in the form

$$x(t) = Xe^{i\omega t} \qquad (3\text{-}19)$$

In general, X and P may be complex numbers. The complex-valued frequency response function for the system is then given by

$$H(i\omega) = \frac{X}{P} \qquad (3\text{-}20)$$

b. *Frequency Response Function of a SDOF system*. Consider an oscillator with mass m, spring k, and damper c, driven by movements of the ground supporting the mass. From the free body diagram the equation of motion in the time domain is

$$m\ddot{x} + c\dot{x} + kx = -m\ddot{y}_g = p(t) \tag{3-21}$$

Assuming $p(t) = Pe^{i\omega t}$, and taking Fourier transform of the above equation, results in the equation of motion in the frequency domain

$$\left(-m\omega^2 + ic\omega + k\right)X(\omega) = P(\omega) \tag{3-22}$$

By definition the complex-valued frequency response function for the SDOF system is given by

$$H(i\omega) = \frac{1}{\left(-m\omega^2 + ic\omega + k\right)} \tag{3-23}$$

or

$$H(i\omega) = \frac{1}{k\left(1 - \beta^2 + 2i\xi\beta\right)}$$

and

$$X(\omega) = H(i\omega)P(\omega) \tag{3-24}$$

b. *Frequency Response Functions for MDOF Systems*. The modal equations of motion for a MDOF system in the frequency domain is expressed by

$$\left[\left(\boldsymbol{K} - \omega^2\boldsymbol{M}\right) + i\left(\omega\boldsymbol{C}\right)\right]\mathbf{Y}(i\omega) = \mathbf{P}(i\omega) \tag{3-25}$$

where $\mathbf{P}(i\omega)$ is the Fourier transform of the modal loading vector $\boldsymbol{P}(t)$, which contains modal components $P_1(t), P_2(t), \ldots, P_n(t)$ as defined in 3-10d; $\mathbf{Y}(i\omega)$ is the Fourier transform of the normal coordinate vector $\mathbf{Y}(t)$ containing $Y_1(t)$, $Y_2(t)$, …, $Y_n(t)$; $\boldsymbol{K}$ and $\boldsymbol{M}$ are the diagonal modal stiffness and mass matrices containing elements in accordance with Equations (3-10b) and (3-10a); and $\boldsymbol{C}$ is the normal modal damping having elements as given by Equation (3-10c). Note that Equation (3-25) may contain all N normal modal equations or only a smaller specified number of lower modes that provide reasonable accuracy. Making use of the impedance matrix $\boldsymbol{I}(i\omega)$ which is equal to the entire bracket matrix, Equation (3-25) can be written in the following compact form

$$\mathbf{Y}(i\omega) = \mathbf{I}(i\omega)^{-1}\mathbf{P}(i\omega) \tag{3-26}$$

From this equation it is obvious that the complex-frequency-response functions $\mathbf{H}(i\omega)$ is the same as inverse of impedance matrix. In practical computation the elements of complex-frequency response functions are obtained from inversion of the impedance matrix, but interpolation procedures are used to reduce the solution efforts. This way frequency response functions are computed at specified number of frequencies but interpolated at the intermediate closely spaced frequency increments.

3-9. Computation of Structural Response

After obtaining all frequency response functions $H_{ij}(i\omega)$ by inversion solution of the impedance matrix and the use of interpolation, the response vector $\mathbf{Y}(i\omega)$ is obtained by superposition using

$$\mathbf{Y}(i\omega) = \mathbf{H}(i\omega)\mathbf{P}(i\omega) \tag{3-27}$$

where $\mathbf{H}(i\omega)$ is the NxN complex-frequency-response matrix

$$\mathbf{H}(i\omega) = \begin{bmatrix} H_{11}(i\omega) & H_{12}(i\omega) & \cdots & H_{1N}(i\omega) \\ H_{21}(i\omega) & H_{22}(i\omega) & \cdots & H_{2N}(i\omega) \\ \vdots & \vdots & \vdots & \vdots \\ H_{N1}(i\omega) & H_{N2}(i\omega) & \cdots & H_{NN}(i\omega) \end{bmatrix} \tag{3-28}$$

determined for an appropriate range of excitation frequencies. Once the complex frequency response matrix has been determined the modal response of structure $\mathbf{Y}(i\omega)$ to any arbitrary loading can be obtained simply by Fourier transforming the load by the FFT procedure and multiplying the results by the complex-frequency-response matrix. Knowing the modal response vector $\mathbf{Y}(i\omega)$, the corresponding modal displacements $\mathbf{Y}(t)$ in the time domain can be obtained by the inverse FFT procedure, and other response quantities are then easily obtained using the standard time-domain procedure described previously.

3-10. Selection of Parameters for Frequency Domain Analysis

To ensure that frequency domain analysis leads to accurate dynamic response of a structure, the parameters that govern the response computation must be selected carefully, as described below.

a. Maximum Excitation Frequency. The maximum excitation frequency selected for the analysis should be greater than the frequencies of all significant harmonics in the input ground acceleration record. The maximum excitation frequencies should also be large enough to include the range of frequencies over which the structure has significant dynamic response.

b. Number of Generalized Coordinates or Vibration Modes. The number of vibration modes required to represent earthquake response of a hydraulic structure is much less than the number of degrees of freedom in the finite-element model. In general, all the modes that contribute significantly to the dynamic response should be included. A final check that enough modes have been considered is to investigate that the maximum stresses and section forces do not change if the number of modes is increased.

c. Number of Excitation Frequencies and Time Interval. As illustrated in Figure 3-1, the parameters used in an FFT analysis include T, Δt, N, f_{max}, and Δf. These are described below.

(1) $T = N^*\Delta t$ is the duration of response history. In Fourier analysis both the excitation and response are periodic; i.e. the values at times … $t\text{-}2T$, $t\text{-}T$, t, $t+T$, $t+2T$ … are the same. Considering that an acceleration record is generally thought to be non-periodic, the input signal should be augmented by several seconds of "trailing zeros" to satisfy this condition. It is expected that adequate zeros are added so that the structure response present at the end of the input record decays to a relatively small value.

(2) Δt is the time increment at which excitation and response values are defined.

(3) $N = T/\Delta t$ is the number of discrete time steps. The choice of N is made so that $N=2^j$ by selecting T as demonstrated in Figure 3-1:

$$T = T_p + T_h$$

where T_p is the duration of the earthquake signal, and T_h is the time required for the structure response to decay to decay to negligible value at the end of earthquake signal.

(4) $f_{max} = 1/2\Delta t$ is the maximum frequency that is included in the analysis and is referred to as the *Nyquist* frequency, the largest frequency contain in the input acceleration record.

(5) $\Delta f = 1/T$ is the frequency increment in Hz. The frequencies included in the analysis are 0, Δf, 2 Δf, 3 Δf, …, f_{max} – a total of $N/2+1$ frequencies.